

The Emergence of Hopf Term and The Marginality of Dirac model

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Abstract

The subtle relation of the marginality of Dirac model and the emergence of the Hopf term through a Yukawa-type interaction is revealed in this work. We show that the improvement of the marginality of Dirac mode through an infinitesimal non-relativistic term, which is irrelevant for renormalization group, is necessary to emerge the Hopf term. It is found that the appearance of the Hopf term is always accompanied with chiral boundary modes, and the sign of the coupling constant of the infinitesimal non-relativistic term decides their chirality. A lattice Dirac model is also constructed to realize the Hopf term.

Keywords:

Quantum anomaly, Marginality of Dirac model, Hopf term, Lattice model

In spatial dimension d , the unit vector field $n(x) \in S^d$, which may be resulted from symmetry breaking, can have soliton excitations due to the homotopy group $\pi_d(S^d) \cong \mathbf{Z}$. For $d > 2$, these solitons can only have fermionic or bosonic statistics because of $\pi_{d+1}(S^d) \cong \mathbf{Z}_2$, while in two spatial dimensions they may have anyonic statistics since $\pi_3(S^2) \cong \mathbf{Z}[1, 2]$. The value of the Hopf term is the corresponding homotopy number for a specific $\mathbf{n}(x)$ in space-time. Thus its coupling constant decides the statistics of the solitons, which are called skyrmions in two spatial dimension. Lots of previous works in both relativistic quantum field theory and condensed matter physics have confirmed that the Hopf term can be emerged out by the Yukawa-type interaction between $\mathbf{n}(x)$ field and fermions, but with coupling constants only for fermionic or bosonic statistics[2, 3, 4, 5, 6, 7, 8, 9].

In the present work, we focus on the emergence of the Hopf term from

the Yukawa-type interaction with Dirac fermions. Although it is proposed that the Hopf term can be emerged out in this way[2, 6, 7, 8, 9], there is a subtleness in the model related to the regularization scheme and irrelevant terms, which can be exactly summed up as the marginality of the Dirac model from our analysis. The emergence of the Hopf term in this relativistic mode is due to a homotopy number in the (ω, \mathbf{k}) -space just like that in models in condensed matter physics[3, 4, 10, 11, 12, 13]. We show that the marginal characteristic of the Dirac model and its improvement play significant roles in the emergence of the Hopf term. Specifically, adding an infinitesimal non-relativistic term, which is irrelevant for renormalization group, is necessary to emerge the Hopf term. Through the connection of this model with the theory of topological insulators[11, 12, 13], we can see that the appearance of the Hopf term is always accompanied with chiral boundary excitations. Moreover the irrelevant and infinitesimal term has physical effects that the sign of its coupling constant can be reflected by the chirality of boundary modes.

We start with the Dirac Hamiltonian in (2+1) dimensions,

$$\mathcal{H}(\mathbf{k}) = -\gamma_0 \boldsymbol{\gamma} \cdot \mathbf{k} + \gamma_0 m,$$

where we choose $\gamma_0 = \tau_3$, $\gamma_1 = i\tau_1$ and $\gamma_2 = i\tau_2$ with τ_i being Pauli matrices, and alternatively,

$$\mathcal{H}(\mathbf{k}) = \tau_2 k_x - \tau_1 k_y + \tau_3 m.$$

The corresponding Green's function, $G(t, \mathbf{r}) = \langle \psi^\dagger(t, \mathbf{r}) \psi(0, 0) \rangle$, can be expressed in (ω, \mathbf{k}) -space and Euclidean signature as

$$G^{-1}(\omega, \mathbf{k}) = i\omega - \mathcal{H}(\mathbf{k}).$$

To introduce the (ω, \mathbf{k}) -space topology, we regard $G^{-1}(\omega, \mathbf{k})$ as a mapping:

$$\begin{aligned} G^{-1} : \quad \mathcal{M} &\rightarrow GL(N, \mathbf{C}) \\ (\omega, \mathbf{k}) &\mapsto G^{-1}(\omega, \mathbf{k}) \end{aligned}$$

with $N = 2$ for Dirac Hamiltonian and \mathcal{M} being the manifold of (ω, \mathbf{k}) -space, and then compactify \mathcal{M} as S^3 by treating the infinity as one point. The smooth mappings from S^3 to $GL(N, \mathbf{C})$ can be classified by the homotopy group $\pi_3(GL(N, \mathbf{C})) \cong \mathbf{Z}$. We can judge which class the $G^{-1}(\omega, \mathbf{k})$ belongs to through computing the homotopy number by the following formula,

$$N[G] = \frac{1}{24\pi^2} \text{tr} \int d\omega d^2k \epsilon^{\mu\nu\rho} G \partial_\mu G^{-1} G \partial_\nu G^{-1} G \partial_\rho G^{-1}, \quad (1)$$

which gives out the homotopy number. However, the infinity is singular for the Dirac model, because \mathcal{H} approaches different matrices when \mathbf{k} is going to the infinity along different directions. The marginal characteristic of the Dirac model lies in the fact that, if we keep plugging $G^{-1} = i\omega - \mathcal{H}$ into Eq.(1), the result is

$$N = -\frac{1}{2} \text{sgn}(m) \quad (2)$$

with $\text{sgn}(a)$ being the sign of a , which is a half integer. To cure this illness at infinity, we modify the Dirac model a little by adding a term that may be sent to zero after all calculations. We may add a term preserving the Lorentz invariance, such as Bk^{2n} with n being a positive integer, for instance,

$$\mathcal{H}_m(\mathbf{k}) = \tau_2 k_x - \tau_1 k_y + \tau_3 (m - Bk^2).$$

As for models like this, although the behavior of the Green's function at infinity is cured, we note that the homotopy number calculated by Eq.(1) is zero. To obtain a nonzero homotopy number, we have to add a non-relativistic term, and for instance modify the Dirac model as

$$\mathcal{H}_M(\mathbf{k}) = \tau_2 k_x - \tau_1 k_y + \tau_3 (m - B\mathbf{k}^2). \quad (3)$$

We may send B to zero after all the calculations. Substituting $G_M^{-1} = i\omega - \mathcal{H}_M$ into Eq.(1), we can obtain

$$N = -\frac{1}{2} (\text{sgn}(m) + \text{sgn}(B)). \quad (4)$$

Actually the above result holds for any term with the form $B\mathbf{k}^{2n}$. For a given m , the crossing over $B = 0$ may send a topologically trivial Green's function to a nontrivial one, or vise versa.

We see that the Dirac model does have the marginality in the sense that an infinitesimal modification of B can send it to a topologically nontrivial one or a trivial one depending on the sign of B , which is illustrated in Fig(1). In our case, the homotopy number has geometric meanings. If we look at the whole parameter space of the Green's function, *i.e.*, the space spanned by $(\omega, \mathbf{k}; B, m)$, there are some points where $G^{-1}(\omega, \mathbf{k})$ is singular. The

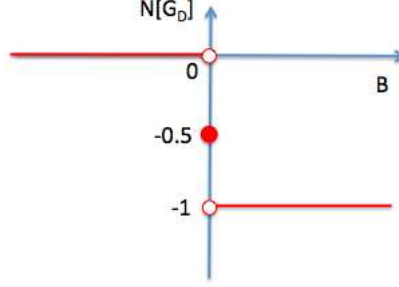


Figure 1: The marginal characteristic of the Dirac model

homotopy number can be nonzero only if (ω, \mathbf{k}) -space as a three-dimensional surface enclose some of these singular points.

Lots of previous works [6, 7, 8, 2, 9] reveal that Hopf term can be emerged out by coupling the Dirac model to a unit-vector field $\mathbf{n}(x)$ (which may be an $O(3)$ -order parameter after symmetry breaking) by a Yukawa-type interaction, *i.e.*,

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \partial_\mu - m\mathbf{n} \cdot \boldsymbol{\sigma}) \psi, \quad (5)$$

where $\mathbf{n}(x) \in S^2$ varies slowly in space-time, and σ_i are pauli matrices acting on the isospin space of ψ . However there exists a subtleness related to the marginal character of the Dirac model, which is not manifestly presented in the previous works. One of main purposes of this work is to elucidate this subtleness. The key trick to deduce the Hopf term is to express $\mathbf{n}(x) \cdot \boldsymbol{\sigma}$ as [3, 7, 10]

$$\mathbf{n}(x) \cdot \boldsymbol{\sigma} = U^\dagger(x) \sigma_3 U(x)$$

with $U(x) \in SU(2)$, and introduce the auxiliary gauge field

$$\Omega_\mu = iU\partial_\mu U^\dagger.$$

The Hopf term can be expressed in terms of the auxiliary gauge field $a_\mu = \langle \mathbf{n} | i\partial_\mu | \mathbf{n} \rangle$ [7, 1], where $|\mathbf{n}\rangle$ being the eigenstate of $\mathbf{n} \cdot \boldsymbol{\sigma}$ with eigenvalue +1, as

$$H_{Hopf} = -\frac{1}{4\pi^2} \int d^3x \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda.$$

Since $|\mathbf{n}\rangle = U^\dagger | \uparrow \rangle$, The Hopf term can also be expressed in terms of $\Omega_\mu = \Omega_\mu^i \sigma^i$ as

$$H_{Hopf} = -\frac{1}{16\pi^2} \int d^3x \epsilon^{\mu\nu\lambda} \Omega_\mu^3 \partial_\nu \Omega_\lambda^3.$$

After making the gauge transformation $\psi \rightarrow U\psi$ and integrating over the fermionic freedoms, we can obtain the Hopf term by dropping higher order terms in the gradient expansion,

$$\begin{aligned}\mathcal{L}_{Hopf} &= -\frac{i}{16\pi}N \int d^3x \epsilon_{\alpha\beta\gamma} \Omega_3^\alpha \partial^\beta \Omega_3^\gamma(x) \\ &= i\pi N[G]H_{Hopf}\end{aligned}\tag{6}$$

where N is given by Eq.(1) and in this case

$$G^{-1} = \begin{pmatrix} i\omega - \mathcal{H}(\mathbf{k}; m) & \\ & i\omega - \mathcal{H}(\mathbf{k}; -m) \end{pmatrix}.$$

In the language of Feynman diagrams, this term comes from two-vertex and three-vertex diagrams.

Since G^{-1} is diagonal, using Eq.(2) we have

$$N[G] = N[G^{00}] + N[G^{11}] = -\frac{1}{2}(\mathbf{sgn}(m) + \mathbf{sgn}(-m)) = 0.$$

Thus exactly speaking, there is no Hopf term emergent from Dirac model without certain terms to improve the marginal characteristic of the Dirac model. Obviously we can use Eq.(3) to do the improvement, however the appearance of Hopf term depends on the way to add the term of $B\mathbf{k}^2$. If we simply substitute m with $m - B\mathbf{k}^2$ in Eq.(5), it is easy to confirm that the homotopy number is still zero using Eq.(4). The right way is to modify Eq.(5) as

$$\mathcal{L}_M = \bar{\psi} \left(i\gamma^\mu \partial_\mu - m\mathbf{n} \cdot \boldsymbol{\sigma} - B\nabla^2 \right) \psi,\tag{7}$$

which corresponds to

$$G_M^{-1} = \begin{pmatrix} i\omega - \mathcal{H}_M(\mathbf{k}; m, B) & \\ & i\omega - \mathcal{H}_M(\mathbf{k}; -m, B) \end{pmatrix}$$

with \mathcal{H} given by Eq.(3). From Eq.(4), we obtain

$$N[G_M] = \mathbf{sgn}(B).$$

Thus an infinitesimal additional term of $B\mathbf{k}^2$ can ensure the emergence of the Hopf term in Eq.(5). This term is irrelevant in the sense of renormalization group, but its sign does have physical meanings when we consider the boundary of the system. In Eq.(5) or Eq.(7), we assume that $\mathbf{n}(x)$ varies slowly

enough in spacetime. So when treating $\mathbf{n}(x)$ as a constant, we see $N[G]$ is just the Chern-number of the corresponding fermionic system as a topological insulator. The nontrivial Chern-number N implies there are N flavors of chiral fermions on the boundary, and the sign of N indicates the chirality of the boundary excitations[5, 14, 11, 12]. As for Eq.(7), if the irrelevant term has positive(negative) coupling constant B , the boundary excitations are left-handed(right-handed) fermions. Thus though different signs of N in Eq.(6) implies the same statistics of the skyrmions, but they imply different chiralities of boundary excitations. Moreover, as the Hopf term is always accompanied with the (ω, \mathbf{k}) -space homotopy number in Eq.(6), the fermionic statistics of skyrmions indicates the chiral boundary excitations.

We carry on to the lattice model. The square lattice version of Dirac model is

$$\begin{aligned} H_0 &= \sum_{\vec{r}, i} \left[\psi_{\vec{r}}^\dagger \left(\frac{c\tau_3 - i\tau_i}{2} \right) \psi_{\vec{r}+\hat{i}} + h.c. \right] + M \sum_{\vec{r}} \psi_{\vec{r}}^\dagger \tau_3 \psi_{\vec{r}} \\ &= \sum_{\mathbf{k}} \psi_{\mathbf{k}}^\dagger \left[\sum_i \sin k_i \gamma^i + \left(M + c \sum_i \cos k_i \right) \gamma^0 \right] \psi_{\mathbf{k}}, \end{aligned}$$

where the two sublattices have different on-site energy, *i.e.*, one has M and the other has $-M$. It describes a topological insulator with the topological number calculated by Eq.(1)[13], that is

$$N = \begin{cases} 1 & 0 < M < 2c \\ -1 & -2c < M < 0 \\ 0 & otherwise \end{cases} \quad (8)$$

Without losing any generality, we have assumed $c > 0$. To obtain the Hopf term from this lattice fermionic model, we add an appropriate coupling term to the Hamiltonian:

$$H_1[\mathbf{n}] = \lambda \mathbf{n}_{\vec{r}} \cdot \boldsymbol{\sigma}$$

with $\lambda > 0$, which may describe a spin interaction. Then the homotopy number as the coupling constant of the Hopf term is calculated by the following Green's function.

$$G_L^{-1}(\omega, \mathbf{k}) = \begin{pmatrix} G_{\uparrow\uparrow}^{-1}(\omega, \mathbf{k}) & \\ & G_{\downarrow\downarrow}^{-1}(\omega, \mathbf{k}) \end{pmatrix},$$

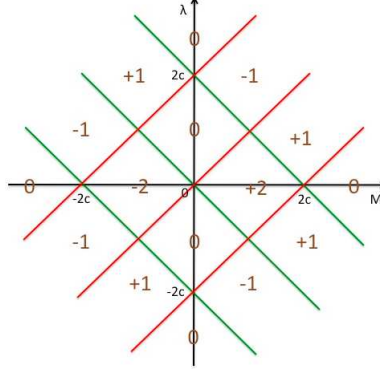


Figure 2: The homotopy number diagram. The black lines are coordinates of the $M - \lambda$ plane. Green and red lines separate the plane into several regions. Each region has its own homotopy number that is written in brown.

where

$$G_{\uparrow\uparrow}^{-1}(\omega, \mathbf{k}) = i\omega - (\sin k_x \tau_x + \sin k_y \tau_y) - \left(M + \lambda + c \sum_i \cos k_i \right) \tau_z$$

and

$$G_{\downarrow\downarrow}^{-1}(\omega, \mathbf{k}) = i\omega - (\sin k_x \tau_x + \sin k_y \tau_y) - \left(M - \lambda + c \sum_i \cos k_i \right) \tau_z$$

Using Eq.(8), we can calculate the homotopy number for a specific pair of M and λ . The result is shown in Fig.[2]. For regions with homotopy number ± 1 , the skyrmions are fermions, while for the others with ± 2 or 0 , the skyrmions are bosons. However, given $\mathbf{n}(x)$ varies slowly enough, a nonzero homotopy number implies chiral boundary modes with chirality being decided by its sign.

To conclude, we illustrated the marginality of the Dirac model, and revealed its relation to the emergence of the Hopf term. We showed that, to emerge the Hopf term from Dirac model, it is necessary to improve the marginality through adding an infinitesimal non-relativistic term, which has

the physical consequence that its sign decides the chirality of boundary modes. We also constructed a lattice Dirac model to emerge the Hopf term.

This work was supported by the GRF (HKU7058/11P) and the CRF (HKU8/11G) of Hong Kong.

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